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An Ambiguity Function Independent of Assumptions About Bandwidth and Carrier Frequency

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An Ambiguity Function Independent of Assumptions About Bandwidth and Carrier Frequency

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Abstract: With the use of a mean-square difference criterion to distinguish echoes from two targets at different ranges moving with different velocities, an ambiguity function is derived. The concept of a modulated carrier is avoided, and the actual Doppler effect of time compression or expansion is used, rather than the more usual approximation of constant frequency shift. Thus this function can be applied to the very-low-frequency broadband signals sometimes employed in sonar systems. It reduces to the Woodward ambiguity function in the case of two targets of nearly equal velocity, and, in general, in a narrow-band approximation. Some properties of this ambiguity function are discussed.

INTRODUCTION

The ambiguity function has been used by Woodward (1) and others (2-7) to study range and velocity resolution of radar and sonar targets. In most of this work, the Doppler effect has been approximated by a frequency shift, which is valid for narrow-band signals. Cahlander (8) has used statistical detection theory to study the wideband signals used by bats for echolocation; Kelly and Wishner (9) have extended the theory to include the effects of high velocity and accelerating targets. Their ambiguity function reduces to that of Woodward for narrow-band signals, if terms of the order of $\exp(\pm 2i\omega_0 t)$ are neglected.

By means of a simplified model for the echo of a signal from a moving target, a mean-square criterion is used to derive an ambiguity function. This function can be used to study the resolution of echoes from two such targets. It is equivalent to the "signal function" of Cahlander (8), although the derivation and application are quite different, and to the ambiguity function of Kelly and Wishner (9). Here, however, neither bandwidth nor carrier frequency enter into the derivation. In the case of sonar signals, the velocity of the target v need not be very small relative to the velocity of propagation of the signal c; all that is required is that $v < c$. By suitable identification of parameters, the new ambiguity function can be identified with Woodward's function in the case of two targets of nearly equal velocities, and in the narrow-band approximation to the detection of echoes from a single target.

THE DOPPLER EFFECT MODEL

Let $s(t)$ be a real-valued function, square integrable on $(-\infty, \infty)$, representing a signal transmitted with a propagation velocity c, assumed constant. Let $r(t)$ be the radial distance and v be the velocity (assumed radial and constant) of a reflecting object. The signal portion of the received (reflected) waveform is then

$$x(t) = a\, s[t - T(t)], \tag{1}$$

where a is a constant (possibly complex to account for phase change on reflection) and $T(t)$ is the time required for the signal to reach the object and return. Thus, a signal received at

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time t was transmitted at time $t - T(t)$, and was incident on the target at time $t - T(t)/2$, when the target position was $r_0 + vt - vT(t)/2$. Hence,

$$cT(t)/2 = r_0 + vt - vT(t)/2$$

or

$$T(t) = \frac{2(r_0 + vt)}{c + v}, \tag{2}$$

and, from this,

$$x(t) = a\, s(\alpha t - T_0), \tag{3}$$

where

$$\alpha = \frac{c - v}{c + v} \tag{4}$$

is the "Doppler stretch factor" and

$$T_0 = \frac{2r_0}{c + v} \tag{5}$$

is the delay of the signal at $t = 0$.

THE AMBIGUITY FUNCTION

If there is another target with radial velocity v' at a radial distance $r'(t) = r_0' + v't$, the signal reflected from it will be

$$x'(t) = a's(\alpha' t - T_0') \tag{6}$$

where the constants are defined as before. We wish to distinguish the waveforms given by Eqs. (3) and (6). Hence, following Woodward (1), we would like their mean-squared difference

$$\int_{-\infty}^{\infty} [x(t) - x'(t)]^2\, dt \tag{7}$$

to be as large as possible for all values of the parameters, except, of course, in a small region near $x(t) = x'(t)$, when the targets are in fact indistinguishable.

Expanding Eq. (7) we get

$$\begin{aligned}
\int_{-\infty}^{\infty} [x(t) - x'(t)]^2\, dt &= \int_{-\infty}^{\infty} x^2(t)\, dt + \int_{-\infty}^{\infty} x'^2(t)\, dt - 2\int_{-\infty}^{\infty} x(t)x'(t)\, dt \\
&= a^2 \int_{-\infty}^{\infty} s^2(\alpha t - T_0)\, dt + a'^2 \int_{-\infty}^{\infty} s^2(\alpha' t - T_0')\, dt \\
&\qquad - 2aa' \int_{-\infty}^{\infty} s(\alpha t - T_0)\, s(\alpha' t - T_0')\, dt \\
&= \frac{\alpha' a^2 + \alpha a'^2}{\alpha\alpha'} - 2aa' \int_{-\infty}^{\infty} s(\alpha t - T_0)\, s(\alpha' t - T_0')\, dt,
\end{aligned} \tag{8}$$

where we assume the signal to be normalized so that

$$\int_{-\infty}^{\infty} s^2(t)\, dt = 1.$$

Since the first term on the right of Eq. (8) does not depend on the signal waveform, and since a and a' may be complex, we see that we can maximize Eq. (7) by choosing a signal which minimizes the modulus of the second term on the right in Eq. (8).

Equivalently, we can define the correlation function

$$\theta(\tau, \gamma) \triangleq \int_{-\infty}^{\infty} s(t)\, s(\gamma t + \tau)\, dt, \tag{9}$$

where either

$$\gamma = \frac{\alpha'}{\alpha} \text{ and } \tau = \frac{\alpha'}{\alpha} T_0 - T_0' \tag{10}$$

or

$$\gamma = \frac{\alpha}{\alpha'} \text{ and } \tau = \frac{\alpha}{\alpha'} T_0' - T_0, \tag{11}$$

and require that $|\theta(\tau, \gamma)|$ be as small as possible except in the vicinity of

$$\theta(0, 1) = \int_{-\infty}^{\infty} s^2(t)\, dt = 1. \tag{12}$$

Equation (12) represents, of course, the correlation of signals reflected from targets at the same range and with the same velocity.

If we define the ambiguity function as

$$\psi(\tau, \gamma) \triangleq |\theta(\tau, \gamma)|^2, \tag{13}$$

then the "distinguishability" criterion requires that $\psi(\tau, \gamma)$ be as small as possible except near $\psi(0, 1) = 1$. The functions $\theta(\tau, \gamma)$ and, hence, $\psi(\tau, \gamma)$ are independent of any assumptions about bandwidth and carrier frequency.

THE RELATIONSHIP TO THE WOODWARD AMBIGUITY FUNCTION

Let $\hat{s}(t)$ be the Hilbert transform of $s(t)$. The "pre-envelope" of $s(t)$ is defined by Dugundji (10) to be

$$z(t) = s(t) + i\, \hat{s}(t). \tag{14}$$

Then $s(t) = Re\{z(t)\}$, and Eq. (9) becomes

$$\begin{aligned}
\theta(\tau, \gamma) &= \int_{-\infty}^{\infty} Re\{z(t)\}\, Re\{z(\gamma t + \tau)\}\, dt \\
&= (1/4) \int_{-\infty}^{\infty} [z(t) + z^*(t)]\, [z(\gamma t + \tau) + z^*(\gamma t + \tau)]\, dt \\
&= (1/2)\, Re\left\{\int_{-\infty}^{\infty} z(t)\, z^*(\gamma t + \tau) + z(t)\, z(\gamma t + \tau)\, dt\right\} \\
&= (1/2)\, Re\{\theta_1(\tau, \gamma) + \theta_2(\tau, \gamma)\},
\end{aligned} \tag{15}$$

where the asterisk denotes the complex conjugate

$$\theta_1(\tau, \gamma) = \int_{-\infty}^{\infty} z(t)\, z^*(\gamma t + \tau)\, dt, \tag{16}$$

and

$$\theta_2(\tau, \gamma) = \int_{-\infty}^{\infty} z(t)\, z(\gamma t + \tau)\, dt. \tag{17}$$

If, following Kelly and Wishner (9), we were to express $s(t)$ as a modulated carrier,

$$s(t) = Re\{z(t)\} = Re\{[z(t)e^{-i\omega_0 t}]\, e^{i\omega_0 t}\}, \tag{18}$$

then $\theta_2(\tau,\gamma)$, Eq. (17), would be related to the terms of order $\exp(\pm 2i\omega_0 t)$ and could be neglected in a narrow-band approximation, as is done by them. We do not need to do this, however, for in fact $\theta_2(\tau, \gamma)$ vanishes identically. If

$$Z(\omega) = \int_{-\infty}^{\infty} z(t) e^{-i\omega t}\, dt$$

is the Fourier transform of $z(t)$, then we can write

$$\theta_2(\tau, \gamma) = \frac{1}{4\pi^2} \iiint_{-\infty}^{\infty} Z(\omega)\, Z(\nu) e^{i(\nu + \gamma\omega)t}\, e^{i\omega\tau}\, dt\, d\nu\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} Z(\omega)\, Z(-\gamma\omega) e^{i\omega\tau}\, d\omega = 0, \tag{19}$$

since

$$Z(\omega) = 0,\ \omega < 0$$

and

$$Z(-\gamma\omega) = 0,\ \omega > 0,\ \text{for } \gamma > 0.$$

(See, for example, Refs. 10 and 11.)

We can write $s(t)$ as in Eq. (18), where ω_0 is now an arbitrary parameter which may be, but does not have to be, associated with a "carrier." If we let $f(t) = z(t) \exp[-i\omega_0 t]$, Eq. (16) becomes

$$\theta_1(\tau, \gamma) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega_0 t}\, f^*(\gamma t + \tau) e^{-i\omega_0(\gamma t + \tau)}\, dt$$

$$= e^{-i\omega_0\tau} \int_{-\infty}^{\infty} f(t)\, f^*(\gamma t + \tau)\, e^{-i\phi t}\, dt$$

$$\approx e^{-i\omega_0\tau}\, \chi(\tau, \phi), \tag{20}$$

where

$$\phi = (\gamma - 1)\omega_0, \tag{21}$$

and

$$\chi(\tau, \phi) = \int_{-\infty}^{\infty} f(t)\, f^*(t + \tau)\, e^{-i\phi t}\, dt \tag{22}$$

is the combined time and frequency correlation function of Woodward (1). The approximation in Eq. (20) holds in the case of two targets of nearly equal velocity. It requires, of course, that $f(t)$ be slowly varying as compared to $\exp(-i\phi t) = \exp[-i(\gamma-1)\omega_0 t]$, but it must be remembered that ω_0 is arbitrary.

If we consider the echo from one target (*i.e.*, let $a' = \alpha' = 1$, $T_0' = 0$), and if we identify ω_0 with a single transmitted frequency, then ϕ is the actual Doppler shift of this single frequency. It is a constant, but no assumptions have been made about the constancy of Doppler shift across a band of frequencies.

From Eqs. (13), (15), (19), and (20), we have

$$\psi(\tau, \gamma) = (1/4)|Re\{\theta_1(\tau, \gamma)\}|^2 \leq (1/4)|\theta_1(\tau, \gamma)|^2 \approx (1/4)\,\psi(\tau, \phi), \tag{23}$$

where ϕ is given by Eq. (21), and

$$\psi(\tau, \phi) = |\chi(\tau, \phi)|^2 \tag{24}$$

is the Woodward ambiguity function.

SOME PROPERTIES OF THIS AMBIGUITY FUNCTION

By analogy with the properties of the Woodward ambiguity function, as discussed by Siebert (12) and others, some of the properties of the new function can be summarized.

If

$$S(\omega) = \int_{-\infty}^{\infty} s(t) e^{-i\omega t}\, dt$$

is the Fourier transform of $s(t)$, we can write for Eq. (9)

$$\begin{aligned}\theta(\tau, \gamma) &= \frac{1}{4\pi^2} \iiint_{-\infty}^{\infty} S(\nu)\, S(\omega)\, e^{i(\nu+\gamma\omega)t}\, e^{i\omega\tau}\, d\omega\, d\nu\, dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, S(-\gamma\omega)\, e^{i\omega\tau}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, S^*(\gamma\omega)\, e^{i\omega\tau}\, d\omega,\end{aligned} \tag{25}$$

since $s(t)$ is real.

By consideration of

$$\int_{-\infty}^{\infty} [s(t) - \lambda s(\gamma t + \tau)]^2\, dt \geq 0 \tag{26}$$

for all λ, and, in particular, Eq. (26) ≥ 0 for $\lambda = \sqrt{\gamma}\,\mathrm{sgn}\,\theta(\tau, \gamma)$, it follows that

$$\theta(\tau, \gamma) \leq \frac{1}{\sqrt{\gamma}}, \text{ for all } \tau, \gamma \tag{27}$$

and

$$\theta(\tau, \gamma) \leq 1, \text{ for all } \tau, \text{ for } \gamma \geq 1. \tag{28}$$

Integration of $\theta^2(\tau, \gamma)$ over all values of its parameters must be limited to at most $0 < \gamma < \infty$, as pointed out by Cahlander (8), who shows, as do Kelly and Wishner (9), that the usual volume constraint on the ambiguity function must be replaced by

$$\int_0^\infty \int_{-\infty}^\infty \theta^2(\tau, \gamma)\, d\tau\, d\gamma = \int_0^\infty \frac{1}{\omega} |S(\omega)|^2\, d\omega. \tag{29}$$

Here $|S(\omega)|^2$ must vanish sufficiently rapidly as $\omega \to 0$ so that the integral in Eq. (29) exists (9). Equation (29) is not independent of signal waveform as is the equivalent integration of the Woodward ambiguity function. In a narrow-band approximation, however, the integral reduces to π/ω_0, a constant (8,9).

In place of the twofold symmetry axis of the Woodward function, we have for $\gamma \neq 0$

$$\theta\left(-\frac{\tau}{\gamma}, \frac{1}{\gamma}\right) = \int_{-\infty}^\infty s(t)\ s\left(\frac{t}{\gamma} - \frac{\tau}{\gamma}\right) dt = \gamma \int_{-\infty}^\infty s(t)\ s(\gamma t + \tau)\ dt$$

$$= \gamma\theta(\tau, \gamma), \tag{30}$$

which shows the relationship between the alternative definitions of the parameters τ and γ, given in Eqs. (10) and (11).

In the case $\gamma = 1$, Eq. (9) becomes

$$\theta(\tau, 1) = \int_{-\infty}^\infty s(t)\ s(t + \tau)\ dt = R(\tau),$$

an autocorrelation function of $s(t)$; and Eq. (16) becomes an autocorrelation function of the "pre-envelope" $z(t)$.

The Woodward ambiguity function is related to the output of a filter matched to a signal $s(t)$, in response to a signal $s(t - \tau) \exp(-i\phi t)$, *i.e.*, an echo with a constant time delay and a constant frequency shift (13). Similarly, the last term on the right of Eq. (8), and hence Eq. (9), is, apart from a constant multiplier, the output of a filter matched to $s(at - T_0)$ when the input is $s(\alpha' t - T_0')$ (9), evaluated at the time the output peak signal power occurs. If we let $a' = \alpha' = 1$ and $T_0' = 0$, then $\theta(\tau, \gamma) = \theta(-T_0, a)$ is the output of a filter matched to the transmitted signal, when the input is an echo from a moving target. Each frequency, ω, in the transmitted signal has been shifted by an amount proportional to itself, *i.e.*, is received as $\omega_d = \alpha\omega$, where α is given by Eq. (4).

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13. ABSTRACT

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